Partially Deamortized Packed-Memory Array

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1 Introduction

One of the deficiencies in the traditional PMA is that one element insertions might trigger a rebalance of the entire array, which costs $\Theta(N)$ element moves. In contrast, the amortized number of element moves, $\Theta(\log^2 N)$, is not that bad. When we do such an insertion in a massive database, triggering a scan of the entire database is infeasible. We may not be able to (or want to) wait while the data structure rebuilds itself. To overcome this deficiency, following the work(s) of Bender, Cole, Demaine, Farach-Colton, and Zito [1], and Willard [4], Haodong Hu in his thesis [2, 3] introduces a partially deamortized packed-memory array whose insert/delete cost per update is at most $\mathcal{O}(\sqrt{N} \log N)$. Even though Haodong Hu's pardially deamortized PMA is considerably simpler than the fully deamortized one introduced by Willard, it is still hard to implement.

In the rest of this paper, we propose an algorithm to partially deamortized the packed-memory that overcomes this deficiency making for a practical implementation to be possible. Our partially deamortized packed-memory array is a cache-oblivious data structure that has a cost per insert of at most $\mathcal{O}(\sqrt{N})$ element moves and $\mathcal{O}(1 + \frac{\sqrt{N}}{B})$ memory transfers, while having the same performance of $\mathcal{O}(1 + \frac{log^2 N}{B})$ amortized memory transfers as the traditional PMA.

We have also used the following referenced to learn about the Packed Memory, Deamortization, and related problems.

1. Eric Demaine's lecture notes [5]
2. Michael Bender's CSE638 lecture notes
3. Jeff Ericsson's Notes on deamortization [6]

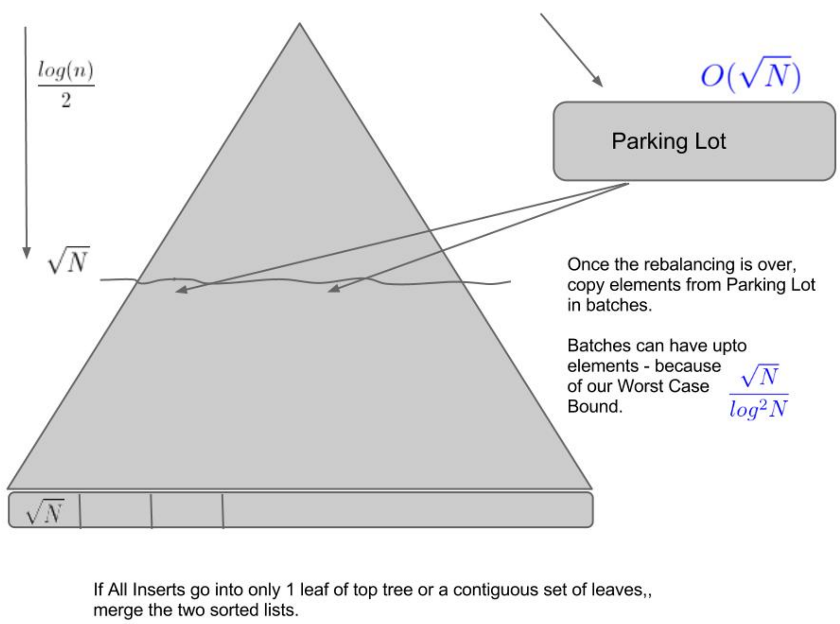
2 How to partially deamortize the PMA?

We shall show how to partially deamortize a PMA (packed memory array) only subject to inserts. Each operation will cost $O(\log^2 n)$ amortized and $O(\sqrt{n})$ worst case.

The general strategy that we shall adopt while trying to partially deamortize the PMA is that of splitting the imaginary tree of the PMA into a top half and a bottom half. The imaginary tree has height $\log_2 n$, so if we split it midway, we get roughly $1 + \sqrt{n}$ imaginary trees, $\sqrt{n}$ of which correspond to actual elements in the PMA. These trees are the $\sqrt{n}$ bottom trees, each having a height of $\frac{\log_2 n}{2}$.

We shall try to achieve the same amortized cost of $O(\log^2 n)$ per element update, but we shall try to bring the worst-case cost per update down from $O(n)$ to $O(\sqrt{n})$. This will come at the cost of $O(n)$ extra space.

If a rebalance in the PMA causes elements to move within one of the bottom trees, we perform such a rebalance normally since it doesn't cost us more than $O(\sqrt{n})$. However, if a rebalance were to affect one of the nodes in the top tree, it most certainly spans more than one bottom tree. In such a situation, we re-build those trees piece-by-piece. While the rebuilding process of these trees is in progress, we shall perform element inserts into another data structure called the *parking lot*.



This *parking lot* can be another PMA or even a simple array. The worst case cost to insert in the PMA or the array is $O(\#\ of\ elements\ in\ the\ parking\ lot)$. We shall show that the parking lot has size $O(\sqrt{n})$ and hence the worst case cost of any insert into the parking lot is $O(\sqrt{n})$. We can not use a linked list since we need to be able to do fast element lookups in the *parking lot*[1].

Theorem 2.1 ***Rebuilding trees without knowing the tree span***

We can rebuild trees without known the span of the tree. i.e. without knowing the number of elements that belong to the current imaginary tree.

Proof: We assume that we need such a tool only to rebalance trees that are rooted in the imaginary top tree, since the bottom trees can be re-balanced normally.

We perform such a rebalance in 2 phases. In phase-1, we start copying elements to a temporary array of size $O(\sqrt{n})$. We shall start copying elements $\sqrt{n}$ at a time.

[1]Thanks to Pablo for pointing this out!

We know that we can choose any window in the PMA as long as the window sizes grow exponentially (include reference - asked as an assignment question, but Dr. Bender mentioned that there is a paper on this). i.e. We select windows of size $\sqrt{n},\ 2\sqrt{n},\ 4\sqrt{n},\ 8\sqrt{n},\ \textit{and so on}\ldots$.

Once we have determined that a certain number of segments (say c) of size $\sqrt{n}$ each is within threshold, we can start copying these elements back to the PMA. The total amortized cost to perform such a rebalance is $O(c\sqrt{n})$. The worst case cost per operation during the whole rebalancing process is $O(\sqrt{n})$.

We note that we need to string together c temporary arrays of size $O(\sqrt{n})$ for this whole operation. Since c is at most $\sqrt{n}$, we need $O(n)$ extra space to perform the rebalance. ■

Theorem 2.2 ***The parking lot has*** $O(\sqrt{N})$ ***elements***

The number of elements in the parking lot never exceeds $O(\sqrt{n})$.

Proof:

The parking will be most full when the root node of our imaginary tree is rebalanced since that will trigger the rebuilding of $O(\sqrt{n})$ lower trees. When the root node is rebalanced, the parking lot will contain $O(\sqrt{n})$ elements. We will show that the parking lot is emptied before the root node is rebalanced again.

In the worst case, the number of elements in the PMA is just 1 less than the maximum allowable for the given size. i.e. the PMA has physical size $2n$, and currently has $n-1$ elements. So, the current density of the PMA is almost 0.5. Before the root node of the PMA is rebalanced, or a new PMA is triggered, it is possible to insert $O(\frac{n}{\log_2 n})$ elements. This is by the standard PMA analysis.

We argue that $O(\sqrt{n})$ is smaller than $O(\frac{n}{\log_2 n})$, i.e the parking lot can be emptied before the root node is touched again.

Note that each of the leaves of size $\Theta(\log_2 n)$ can become completely full since their density threshold is 1.0. Similarly, for the lower trees (of size $\sqrt{n}$ each), the density threshold is roughly 0.75. Hence, it has space for $0.25\sqrt{n}$ elements if we do not violate density thresholds. If we are okay with violating density thresholds (as we shall see later), we can actually insert $\frac{\sqrt{n}}{2}$ elements in each subtree of size $\sqrt{n}$ and hence $\sqrt{n}$ elements in k such subtrees. i.e. If we want to insert $\sqrt{n}$ elements, we set $k=2$. In our case, we would want to insert $2\sqrt{n}$ elements to get $k=4$.

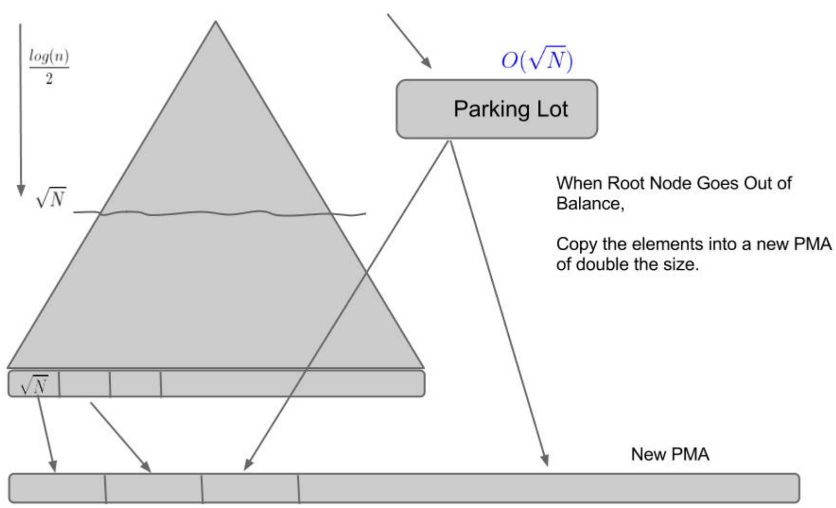
We consider multiple cases:

1. Suppose all the elements in the parking lot go to one of the ends (begin or end) of the PMA or to some specific part of the PMA (a.k.a. *Hammer Inserts*). In this case, we can identify the $2\sqrt{n}$ lower trees that can be rebuilt and we use a simple merging strategy to offload the parking lot into these 2

subtrees. The density thresholds for these subtrees is violated, but that is okay since the next rebuild will take care of that.

2. Suppose every element goes into a different subtree. In this case too, we pay a worst-case cost of $\Theta(\log^2 n)$ per insert (since we have just rebalanced the PMA), and we can actually perform batch inserts of $\Theta\left(\frac{\sqrt{n}}{\log^2 n}\right)$ elements at a time to get a worst case running time of $O(\sqrt{n})$ per operation.

In a similar fashion, we can sub-divide the inserts from the parking lot into the PMA into cases where we insert $\Theta(\sqrt{n})$ elements into a lower tree or fewer and handle them separately.



Additionally, if we detect that the sum of the sizes of the PMA and the parking lot exceeds the density threshold at the root node, we can start copying elements into a new PMA of twice the size. This operation again copies $\sqrt{n}$ subtrees, each of size $O(\sqrt{n})$. Since we do not insert elements into the PMA while it is being rebalanced or copied, our parking lot is still capped at $O(\sqrt{n})$ in size.

■

3 Querying the Partially Deamortized PMA

Querying the *PDPMA* is very simple. If no rebuild is in progress, we query the PDPMA as we normally would. On the other hand, if a rebuild is in progress, we query both the PDPMA as well as the parking lot to answer queries. We are sure that the lower density thresholds are met even during a rebuild since we are making the PMA more sparse only if it has become too dense. i.e. we never violate the lower density thresholds even while we are in the middle of a rebuild operation.

References

[1] Two Simplified Algorithms for Maintaining Order in a List, Michael A. Bender, Richard Cole, Erik D. Demaine, Martin Farach-colton, Jack Zito, 2002

[2] Cache-Oblivious Data Structures for Massive Data Sets, H. Hu, 2007

[3] An Adaptive Packed-Memory Array, M.A. Bender and H. Hu, Transactions on Database Systems Special Issue on PODS '06, 32(4): 2007.

[4] A density control algorithm for doing insertions and deletions in a sequentially ordered file in good worst-case time, D. E. Willard Information and Computation, 97(2):150204, 1992.

[5] http://courses.csail.mit.edu/6.851/spring12/lectures/L08.pdf

[6] http://compgeom.cs.uiuc.edu/~jeffe/teaching/datastructures/notes/01-statictodynamic.pdf